ENGINEERING MATH - II

UNIT 3

LAPLACE TRANSFORM

Vibha Masti

Feedback/corrections: vibha@pesu.pes.edu

- To solve a differential equation, Laplace applied a transformation.
- Developed Fourier transform, R, Z after this

Transformation

1. LVIII + XXX VII → Arabic → add → Roman

2. $147321572 \times 31317895210$ → $\log(a) + \log(b)$ → antilog

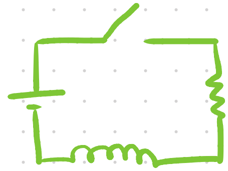
3. $\int \sin(x^2)\, x\,dx$ → $\frac{1}{2}\int \sin y\, dy$ → $-\frac{1}{2}\cos y + C$ → $-\frac{1}{2}\cos x^2 + C$

IN GENERAL,

input → $\boxed{\text{trans}}$ → intermediate → $\boxed{\text{inverse}}$ → output

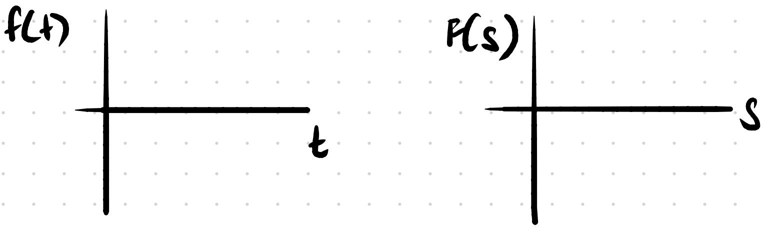
Consider DE

$$\frac{di}{dt} + \int \frac{e^{it}}{t}\,dt + \sin t = 0$$

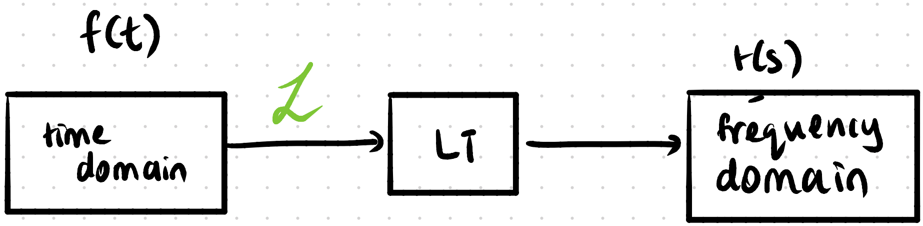


In Laplace Transform, all DE's are converted to algebraic expressions

$$f(t) \longrightarrow F(s)$$ ← easier to handle



eg: signal transmission, ECG



Integral transform

An improper integral =

$$\int_{-\infty}^{\infty} k(s, x)\, f(x)\, dx$$

kernel

is called an integral transform of $f(x)$ if the integral is finite

$k(s,x)$ is called the kernel of the transformation.

Laplace Transform

If $f(t)$ meets the following conditions

1. Should be piecewise continuous
   - ↳ cannot have infinitely many subintervals
   - ↳ should be continuous in each interval

eg: $f(t) = \begin{cases} \sin(t) & -\infty < t < 0 \\ t^2 & 0 \le t \le 3^{10} \\ \cos t & t > 3^{10} \end{cases}$ not continuous at $0, 3^{10}$; but finite

note: $\tan t$ is invalid

2. $f(t)$ is of exponential order s, where s is any complex parameter

3. $\lim_{t \to \infty} e^{-t} f(t)$ = finite

Laplace transform of f(t)

complex parameter $s = \sigma + i\omega$

Defined as $$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = F(s)$$

kernel

new function

(-ve time is meaningless)

$s > 0$

Properties of LT

1. Linearity property

$$\mathcal{L}\{af(t) \pm bg(t)\} = aF(s) + bG(s)$$

2. Scaling Property

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} e^{-st} f(at)\,dt \qquad \begin{aligned} at &= u \\ a\,dt &= du \end{aligned}$$

$$= \frac{1}{a}\int_0^{\infty} e^{-\frac{s}{a}u} f(u)\,du$$

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$ — frequency scaling

and

$$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$$

Find $\mathcal{L}$ of the following common functions

1. $\mathcal{L}\{k\} = \int_0^\infty e^{-st} k \, dt = k\int_0^\infty e^{-st} dt = \frac{k}{s}\left[-e^{-st}\right]_0^\infty$

$= \frac{k}{s}$

2. $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt$

$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty$ if $s > a$ and s is real or s is complex

$= \frac{1}{a-s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

3. $\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(a+s)t} dt$

$= \frac{1}{s+a}$

4. $\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$

$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{+1}{a+s}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{+a}-s\,\cancel{+a}-s}{a^2-s^2}\right) = \frac{s}{s^2-a^2}$$

5. $\mathcal{L}\{\sin at\} = \displaystyle\int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$$

6. $\mathcal{L}\{a^t\} = \mathcal{L}\{e^{\ln a t}\} = \dfrac{1}{s-\ln a}$

7. $\mathcal{L}\{\sin at\} = \mathrm{Im}(\mathcal{L}\{e^{iat}\}) = \mathrm{Im}\left(\dfrac{1}{s-ia}\right)$

$$= \mathrm{Im}\left(\frac{s+ia}{s^2+a^2}\right) = \frac{a}{s^2+a^2}$$

8- $\mathcal{L}\{\cos at\} = \mathrm{Re}(\mathcal{L}\{e^{iat}\}) = \mathrm{Re}\left(\dfrac{s+ia}{s^2+a^2}\right)$

$$= \frac{s}{s^2+a^2}$$

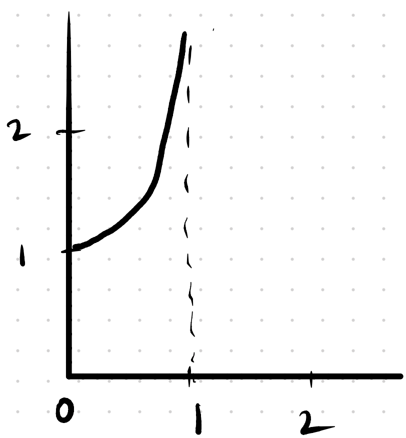
9. $\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$ $\qquad u = st$, $\frac{du}{s} = dt \Rightarrow t = \frac{u}{s}$

$$= \frac{1}{s}\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n du$$

$$= \frac{1}{s^{n+1}}\Gamma(n+1), \quad n \in \mathbb{R} - \{\mathbb{Z}^-\}$$

1. Find the Laplace Transform of the function

$$f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$



$\mathcal{L}\{f(t)\}$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} e^t \, dt + 0$$

$$= \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1$$

rational function $\rightarrow$ $= \dfrac{e^{1-s} - 1}{1-s}$ $\qquad s \neq 1$

2. Find $\mathcal{L}\{t + e^{3t} + 5 + \sinh 3t + 3^t\}$

$$= \frac{1}{s^2} + \frac{1}{s-3} + \frac{3}{s^2-9} + \frac{1}{s-\ln 3}$$

3. Find $\mathcal{L}\{\sinh(at+b)\}$

$$= \int_0^\infty e^{-st} \sinh(at+b)\, dt$$

$$= \mathcal{L}\left\{\frac{e^{-st} e^{at+b} - e^{-st} e^{-at-b}}{2}\right\}$$

$$= e^b \mathcal{L}\left\{\frac{e^{-st} e^{at}}{2}\right\} - e^{-b} \mathcal{L}\left\{\frac{e^{-st} e^{-at}}{2}\right\}$$

$$= \frac{1}{2} e^b \frac{1}{s-a} - \frac{1}{2} e^{-b} \frac{1}{s+a}$$

$$= \frac{e^b}{2(s-a)} - \frac{e^{-b}}{2(s+a)}$$

$$= \frac{1}{2}\left(\frac{s(e^b - e^{-b}) + a(e^b + e^{-b})}{s^2 - a^2}\right)$$

$$= \frac{s}{s^2-a^2} \sinh b + \frac{a}{s^2-a^2} \cosh b$$

4. $\mathcal{L}\{\cosh(at+b)\}$

$$= \mathcal{L}\{\cosh at \cosh b + \sinh ab \sinh b\}$$

$$= \cosh b\, \mathcal{L}\{\cosh at\} + \sinh b\, \mathcal{L}\{\sinh at\}$$

$$= \cosh b \frac{s}{s^2-a^2} + \sinh b \frac{a}{s^2-a^2}$$

5. $\mathcal{L}\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\}$

$$= \mathcal{L}\left\{t^{3/2} + 3t^{1/2} + 3t^{-1/2} + t^{-3/2}\right\}$$

$$= \frac{\Gamma(5/2)}{s^{5/2}} + \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{3\Gamma(1/2)}{s^{1/2}} + \frac{\Gamma(-1/2)}{s^{-1/2}}$$

$$= \sqrt{\pi}\left(\frac{3/2 \times 1/2}{s^{5/2}} + \frac{3 \cdot 1/2}{s^{3/2}} + \frac{3}{s^{1/2}} \frac{-2}{s^{-1/2}}\right)$$

$$= \sqrt{\pi}\left(\frac{3}{4s^{5/2}} + \frac{3}{2s^{3/2}} + \frac{3}{\sqrt{s}} - 2\sqrt{s}\right)$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} + \frac{3}{2s} + 3 - 2s\right)$$

6. $\mathcal{L}\{e^{-4t+5}\} = \dfrac{e^5}{s+4}$

7. $\mathcal{L}\{t^5+4t^4-3t^3+2t^2+t+1\}$

$$= \frac{\Gamma(6)}{s^6} + 4\frac{\Gamma(5)}{s^5} - \frac{3\,\Gamma(4)}{s^4} + \frac{2\,\Gamma(3)}{s^3} + \frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{5!}{s^6} + \frac{4\times 4!}{s^5} - \frac{3\times 3!}{s^4} + \frac{2\times 2!}{s^3} + \frac{1}{s^2} + \frac{1}{s}$$

8. $\mathcal{L}\{\cos(3t+5)\}$

$$= \mathcal{L}\{\cos 3t \cos 5 - \sin 3t \sin 5\} = \frac{(\cos 5)\,s}{s^2+9} - \frac{(\sin 5)\,3}{s^2+9}$$

9. $\mathcal{L}\{5^{-3t}\} = \mathcal{L}\{e^{-3\ln 5\, t}\} = \dfrac{1}{s+3\ln 5}$

10. $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\} = \mathcal{L}\left\{\dfrac{t}{t} - \dfrac{t^3/3!}{t} + \dfrac{t^5/5!}{t} \cdots\right\}$

$$= \mathcal{L}\left\{1 + \frac{-t^2}{3!} + \frac{t^4}{5!} \cdots\cdots\right\}$$

$$= \frac{1}{s^2} - \frac{2!}{3!\,s^3} + \frac{4!}{5!\,s^4} \cdots\cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^3} + \frac{1}{5s^4} + \cdots$$

11. $\mathcal{L}\{\cos 3t \cos 2t \cos t\}$

$$= \mathcal{L}\left\{\frac{1}{2}(\cos 5t + \cos t)\cos t\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\frac{1}{2}(\cos 6t + \cos 4t) + \frac{1}{2}(\cos 2t + 1)\right\}$$

$$= \frac{1}{4}\mathcal{L}\{\cos 6t + \cos 4t + \cos 2t + 1\}$$

$$= \frac{1}{4}\left(\frac{s^2}{s^2+6^2} + \frac{s^2}{s^2+4^2} + \frac{s^2}{s^2+2^2} + \frac{1}{s^2}\right)$$

12. $\mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\left\{\frac{\sqrt{t}}{1!} - \frac{t\sqrt{t}}{3!} + \frac{t^2\sqrt{t}}{5!} - \cdots\right\}$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \cdots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}}\left(1 - \frac{1}{3!}\frac{\times 3/2}{s} + \frac{1}{5!}\frac{\times 5/2 \times 3/2}{s^2} \cdots\right)$$

$$= \frac{\Gamma(3/2)}{s^{3/2}}\left(1 - \frac{1}{2^2 s} + \frac{1}{2^5 s^2} - \frac{1}{7!}\frac{\times 7/2 \times 5/2 \times 3/2}{s^3} + \cdots\right)$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{s\sqrt{s}}\left(1 - \frac{1}{2^2 s} + \frac{1}{2^5 s^2} - \frac{1}{(2^3)(6\times 4\times 2)s^3} + \cdots\right)$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\left(1 - \frac{1}{4s} + \frac{(1/4s)^2}{2!} - \frac{(1/4s)^3}{3!} + \cdots\right)$$

$$= \frac{1}{2s}\sqrt{\frac{\pi}{s}}\ e^{-1/4s}$$

Properties of Laplace Transforms

3. First Shifting Property

$$\mathcal{L}\{f(t)\} = F(s) \text{ , then}$$

$$\mathcal{L}\{e^{\pm at} f(t)\} = F(s \mp a)$$

Proof

$$\int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

13. $\mathcal{L}\{e^{\pm at} \sinh bt\} = \dfrac{b}{(s \mp a)^2 - b^2}$

14. $\mathcal{L}\{e^{\pm at} \cosh bt\} = \dfrac{(s \mp a)}{(s \mp a)^2 - b^2}$

15. $\mathcal{L}\{e^{\pm at} \sin bt\} = \dfrac{b}{(s \mp a)^2 + b^2}$

16. $\mathcal{L}\{e^{\pm at} \cos bt\} = \dfrac{(s \mp a)}{(s \mp a)^2 + b^2}$

17. $\mathcal{L}\{e^{\pm at}\ b^t\} = \dfrac{1}{(s \mp a) - \ln b}$

18. $\mathcal{L}\{e^{\pm at} t^n\} = \dfrac{\Gamma(n+1)}{(s \mp a)^{n+1}}$

19. $\mathcal{L}\{t e^{at}\} = \dfrac{1}{(s-a)^2}$

20. $\mathcal{L}\{t^2 e^{at}\} = \dfrac{2!}{(s-a)^3}$

4. Multiplication by t

If $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$$

* $f(t)$ is NOT exponential

Proof (induction)

if $n=1$

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds} F(s)$$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\frac{d}{ds} F(s) = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$-\frac{d}{ds}F(s) = \mathcal{L}\{t f(t)\}$$

Assume true for $n = k$

$$\frac{d^k}{ds^k}(F(s)) = \mathcal{L}\{t^k f(t)\}$$

For $n = k+1$

$$(-1)^k \frac{d}{ds}\left(\frac{d^k}{ds^k}(F(s))\right) = \frac{d}{ds}\int_0^\infty e^{-st}\, t^k f(t)\, dt$$

$$= \int_0^\infty -1\, t e^{-st}\, t^k f(t)\, dt$$

$$= -1 \int_0^\infty e^{-st}\, t^{k+1} f(t)\, dt$$

$$(-1)^{k+1}\frac{d^{k+1}}{ds^{k+1}}F(s) = \mathcal{L}\{t^{k+1} f(t)\}$$

$$\therefore \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$$

21. $\mathcal{L}\{\sinh^2 3t\}$

$$= \mathcal{L}\left\{\frac{(e^{3t}-e^{-3t})^2}{4}\right\}$$

$$= \frac{1}{4}\mathcal{L}\{e^{6t}+e^{-6t}-2\} = \frac{1}{4}\left(\frac{1}{s-6}+\frac{1}{s+6}-\frac{2}{s}\right)$$

22. $\mathcal{L}\{(\sqrt{e^{-t}}+1)^2\}$

$$= \mathcal{L}\{e^{-t}+1+2e^{-t/2}\} = \frac{1}{s+1}+\frac{1}{s}+\frac{2}{s+1/2}$$

$$= \frac{1}{s+1}+\frac{1}{s}+\frac{4}{2s+1}$$

23. $\mathcal{L}\{|t|\}$

$$|t| = \begin{cases} t, & x \geq 0 \\ -t, & x < 0 \end{cases} \leftarrow \text{integral 0 to } \infty$$

$$= \mathcal{L}\{t\} = \frac{1}{s^2}$$

24. $\mathcal{L}\{t^{-a}\} = \dfrac{\Gamma(-a+1)}{s^{1-a}} \quad (a \notin \mathbb{Z}^+)$

25. $\mathcal{L}\{e^{-5t}\sin 4t\} = \dfrac{4}{(s+5)^2+16}$

26. $\mathcal{L}\{e^{3t}\cosh 2t\} = \dfrac{s-3}{(s-3)^2-4}$

27. Evaluate $\displaystyle\int_0^\infty e^{-2t}\sin 3t \sin 5t\, dt$

$$= \frac{1}{2}\int_0^\infty e^{-2t}(\cos 2t - \cos 8t)\, dt$$

$$= \frac{1}{2}\mathcal{L}\{\cos 2t\}_{s=2} - \frac{1}{2}\mathcal{L}\{\cos 8t\}_{s=2}$$

$$= \frac{1}{2}\frac{s}{s^2+4} - \frac{1}{2}\frac{s}{s^2+64}$$

$$= \frac{1}{2}\frac{2}{8} - \frac{1}{2}\frac{2}{68}$$

5. Division by t

If $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$ iff the integral converges

Proof

$$\int_s^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds = \int_s^\infty F(s)\,ds$$

$$\int_0^\infty \left(\int_s^\infty e^{-st} f(t)\,ds\right) dt$$

$$= \int_0^\infty \left.\frac{e^{-st}}{-t}\right|_s^\infty f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt = \int_0^\infty F(s)\,ds$$

Note: $$\mathcal{L}\left\{\frac{f(t)}{t^n}\right\} = \underbrace{\int_s^\infty \int_s^\infty \cdots \int_s^\infty}_{n \text{ times}} F(s)\,ds \cdots ds$$

iff the integral converges in every step

LT of Derivatives

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Proof

$$= \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\quad v = f(t)$

$du = -se^{-st}$ $\quad dv = f'(t)$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\,F(s)$$

$\lim\limits_{t \to \infty} e^{-st} f(t) = 0$

$$= 0 - f(0) + s\,F(s)$$

$$= s\,F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = -f'(0) + s\mathcal{L}\{f'(t)\}$$

$$= -f'(0) - sf(0) + s^2F(s)$$

In general,

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

LT of Integrals

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

$$\mathcal{L}\left\{\underbrace{\int_0^t\int_0^t \cdots \int_0^t}_{n \text{ times}} f(t)\,dt \dots dt\,dt\right\} = \frac{F(s)}{s^n}$$

Proof

$$g(t) = \int_0^t f(t)\,dt$$

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - \cancel{g(0)}^{\,0}$$

$$\mathcal{L}\{f(t)\} = s\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\left\{\int_0^t f(t)\right\} = \frac{F(s)}{s}$$

28. $\mathcal{L}\{e^{2t}(3\sin 4t - 4\cos 4t)\}$

$$= 3\left(\frac{b}{(s-a)^2-b^2}\right) - 4\left(\frac{s+a}{(s-a)^2-b^2}\right)$$

$$= 3\,\frac{4}{(s-2)^2-4^2} - \frac{4\,(s-2)}{(s-2)^2-4^2}$$

$$= \frac{20-4s}{(s-2)^2-16}$$

29. $\mathcal{L}\left\{\left(\frac{t^{n-1}}{1-e^{-t}}\right)\right\} = \mathcal{L}\{(1-e^{-t})^{-1}t^{n-1}\}$

$$= \mathcal{L}\{(1+e^{-t}+e^{-2t}+e^{-3t}+\dots)t^{n-1}\}$$

$$= \mathcal{L}\{t^{n-1}+e^{-t}t^{n-1}+e^{-2t}t^{n-1}+\dots\}$$

$$= \frac{\Gamma(n)}{s^n} + \frac{\Gamma(n)}{(s+1)^n} + \frac{\Gamma(n)}{(s+2)^n} + \dots$$

$$= \sum_{i=0}^{\infty} \frac{\Gamma(n)}{(s+i)^n}$$

30. $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \mathcal{L}\{\sin t\}\, ds$

$= \int_s^{\infty} \frac{1}{s^2+1}\, ds = \tan^{-1}(s)\Big]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}s$

$= \cot^{-1}(s)$

31. $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$ $\quad f(t) = \sin\sqrt{t}$

$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$

$= 2\mathcal{L}\{f'(t)\} = 2(sF(s) - f(0))$

$= 2(s\,\mathcal{L}\{\sin\sqrt{t}\} - \sin 0)$

$= \cancel{2}(\cancel{s}\,\frac{1}{\cancel{2s}}\sqrt{\frac{\pi}{s}}\, e^{-1/4s})$

$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

32 $$\mathcal{L}\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(a(-1)(s^2+a^2)^{-2}\cdot 2s\right)$$

$$= \frac{d}{ds}\left(\frac{-2as}{(s^2+a^2)^2}\right) = -2a\left(\frac{1}{(s^2+a^2)^2} + \frac{s(-2)\cdot 2s}{(s^2+a^2)^3}\right)$$

$$= -2a\left(\frac{(s^2+a^2) - 4s^2}{(s^2+a^2)^3}\right) = \frac{(a^2-3s^2)(-2a)}{(s^2+a^2)^3}$$

$$= \frac{6as^2-2a^3}{(s^2+a^2)^3}$$

OR

$$\operatorname{Im}\mathcal{L}\{t^2 e^{iat}\} = \operatorname{Im}\left((-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s-ia}\right)\right)$$

$$= \operatorname{Im}\left(\frac{d}{ds}\left((-1)(s-ia)^{-2}\right)\right)$$

$$= \operatorname{Im}\left((-1)(-2)(s-ia)^{-3}\right) = \operatorname{Im}\left(\frac{2}{(s-ia)^3}\right)$$

$$= \operatorname{Im}\left(\frac{2(s+ia)^3}{(s^2+a^2)^3}\right)$$

$$= \frac{2}{(s^2+a^2)^3} \operatorname{Im}\left(s^3 + 3s^2(ia) + 3s(-a^2) - ia^3 \right)$$

$$= \frac{6s^2a - 2a^3}{(s^2+a^2)^3}$$

33. $\mathcal{L}\{ t \cosh at \} = \frac{-d}{ds}\left(\frac{s}{s^2-a^2} \right)$

$$= -\left(\frac{(1)(s^2-a^2) - (s)(2s)}{(s^2-a^2)^2} \right)$$

$$= -\left(\frac{s^2-a^2+2s^2}{(s^2-a^2)^2} \right) = \frac{s^2+a^2}{(s^2-a^2)^2}$$

34. $\mathcal{L}\left\{ \int_0^t \int_0^t \int_0^t \cos au \, du \, du \, du \right\} = \left(\frac{s}{s^2+a^2} \right) \frac{1}{s^3}$

35. $\mathcal{L}\left\{ e^{-4t} \int_0^t \frac{\sin 3t}{t} \, dt \right\} = F_1(s+4)$

$$\mathcal{L}\left\{ \int_0^t \frac{\sin 3t}{t} \, dt \right\} = F_1(s) = \frac{1}{s} F_2(s)$$

$$F_2(s) = \mathcal{L}\left\{ \frac{\sin 3t}{t} \right\} = \int_s^\infty F_3(s) \, ds$$

$$F_3(s) = \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$F_2(s) = \int_s^{\infty} \frac{3}{s^2+3^2}\, ds = \frac{3}{3}\tan^{-1}\left(\frac{s}{3}\right)$$

$$= \tan^{-1}\left(\frac{s}{3}\right)\Big]_0^{\infty} = \cot^{-1}\left(\frac{s}{3}\right)$$

$$F_1(s) = \frac{1}{s}\cot^{-1}\left(\frac{s}{3}\right)$$

$$F(s) = \frac{1}{s+4}\cot^{-1}\left(\frac{s+4}{3}\right)$$

36. $\mathcal{L}\left\{\cosh t \int_0^t e^t \cosh t\, dt\right\}$

$$\mathcal{L}\left\{\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^t \cosh t\, dt\right\}$$

$$= \frac{1}{2}F_1(s-1) + \frac{1}{2}F_1(s+1)$$

$$F_1(s) = \mathcal{L}\left\{\int_0^t e^t \cosh t\, dt\right\} = \frac{F_2(s)}{s}$$

$$F_2(s) = \frac{(s-1)}{(s-1)^2-1^2}$$

$$F_1(s) = \left(\frac{(s-1)}{(s-1)^2-1}\right)\frac{1}{s}$$

$$= \frac{1}{2}\,\frac{(s-2)}{(s-2)^2-1}\cdot\frac{1}{s-1} + \frac{1}{2}\,\frac{s}{s^2-1}\cdot\frac{1}{s+1}$$

37. $$F(s) = \mathcal{L}\left\{\int_0^t e^{-t}\cosh t\,dt\right\}$$

$$= \frac{1}{s}\left(\frac{(s+1)}{(s+1)^2-1^2}\right)$$

38. $$\mathcal{L}\{t(3\sin 2t - 2\cos 2t)\}$$

$$= (-1)\frac{d}{ds}\left(\mathcal{L}\{3\sin 2t - 2\cos 2t\}\right)$$

$$= (-1)\frac{d}{ds}\left(3\left(\frac{2}{s^2+4}\right) - 2\left(\frac{s}{s^2+4}\right)\right)$$

$$= (-1)\left(6(-1)(2s)(s^2+4)^{-2} - 2\left(\frac{1}{s^2+4} + \frac{s(-1)\,2s}{(s^2+4)^2}\right)\right)$$

$$=(-1)\left(\frac{-12s}{(s^2+4)^2} - 2\left(\frac{s^2+4-2s^2}{(s^2+4)^2}\right)\right)$$

$$=(-1)\left(\frac{-12s}{(s^2+4)^2} - \frac{2(4-s^2)}{(s^2+4)^2}\right)$$

$$= \frac{12s+8-2s^2}{(s^2+4)^2}$$

39. $\mathcal{L}\{t^3\cos t\} = (-1)^3 \frac{d^3}{ds^3}\left(\mathcal{L}\{\cos t\}\right)$

$$= -\frac{d^3}{ds^3}\left(Re\left(\mathcal{L}\{e^{it}\}\right)\right)$$

$$= Re\left(-\frac{d^3}{ds^3}\left(\frac{1}{s-i}\right)\right) = Re\left(\frac{d^2}{ds^2}\left((-1)(s-i)^{-2}\right)\right)$$

$$= Re\left((-1)(-2)(-3)(s-i)^{-4}\right)$$

14641

$$= Re\left(\frac{-6(s+i)^4}{(s^2+1)^4}\right) = \frac{-6}{(s^2+1)^4}(s^4-6s^2+1)$$

40. $\mathcal{L}\left\{ \frac{\sin 3t \cos t}{t} \right\} = \frac{1}{2} \mathcal{L}\left\{ \frac{\sin(4t) + \sin 2t}{t} \right\}$

$= \frac{1}{2} \int_s^{\infty} F_1(s)\, ds \; ; \; F_1(s) = \mathcal{L}\left\{ \sin 4t + \sin 2t \right\}$

$F_1(s) = \frac{4}{s^2+4^2} + \frac{2}{s^2+2^2}$

$F(s) = \int_s^{\infty} \frac{4}{s^2+4^2} + \frac{2}{s^2+2^2}\, ds$

$= \frac{1}{2} \cot^{-1}\left(\frac{s}{4}\right) + \frac{1}{2} \cot^{-1}\left(\frac{s}{2}\right)$

41. Show that $\int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$

$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$= \frac{1}{2} \int_0^{\infty} \frac{1-\cos 2t}{t^2}\, dt = \frac{1}{2} \mathcal{L}\left\{ \frac{1-\cos 2t}{t^2} \right\}_{s=0}$

$= \frac{1}{2} \int_s^{\infty} \int_s^{\infty} \frac{1}{s} - \frac{s}{s^2+4}\, ds\, ds$

$$= \frac{1}{2}\int_s^\infty [\ln s]_s^\infty - \frac{1}{2}[\ln(s^2+4)]_s^\infty \, ds$$

$$= \frac{1}{2}\int_s^\infty \ln\left(\frac{s}{\sqrt{s^2+4}}\right)\Bigg|_s^\infty ds = \frac{1}{4}\int_s^\infty \ln\left(\frac{s^2}{s^2+4}\right)\Bigg|_s^\infty ds$$

$$= \frac{1}{4}\int_s^\infty \lim_{s\to\infty}\left(\ln\left(\frac{s^2}{s^2+4}\right)\right) - \ln\left(\frac{s^2}{s^2+4}\right) ds$$

$\ln 1 = 0$

$$= \frac{1}{4}\int_s^\infty \lim_{s\to\infty} \ln\left(1 - \frac{4}{s^2+4}\right) - \ln\left(\frac{s^2}{s^2+4}\right) ds$$

$$= -\frac{1}{4}\int_s^\infty \ln\left(\frac{s^2}{s^2+4}\right) ds = \frac{-1}{2}\int_s^\infty \left(\ln s - \ln\left(\sqrt{s^2+4}\right)\right) ds$$

$u = \ln s - \ln\left(\sqrt{s^2+4}\right)$ $\quad v = s$

$du = \frac{1}{s} - \frac{\cancel{2}s}{\cancel{2}(s^2+4)}$ $\quad dv = ds$

$= \frac{1}{s} - \frac{s}{s^2+4}$

$$= \frac{-1}{2}\left(\left[s\ln s - s\ln\left(\sqrt{s^2+4}\right) \right]_s^{\infty} - \int_s^{\infty} 1 - \frac{s^2}{s^2+4}\, ds \right)$$

$$= \frac{-1}{2}\left(\left[s\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^{\infty} - \int_s^{\infty} \frac{4}{s^2+4}\, ds \right)$$

$$= \frac{-1}{2}\left(\lim_{s\to\infty} s\ln\left(\frac{1}{\sqrt{1+4/s^2}}\right) - s\ln\left(\frac{s}{\sqrt{s^2+4}}\right) - 2\tan^{-1}\left(\frac{s}{2}\right)\Big|_s^{\infty} \right)$$

$$= \frac{+1}{2}\left(\lim_{s\to\infty} \frac{s}{2}\ln\left(1+\frac{4}{s^2}\right)\right) + \frac{1}{2}s\ln\left(\frac{s}{\sqrt{s^2+4}}\right) + \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)\right)$$

$$= \frac{1}{2}\left(\lim_{s\to\infty} \frac{s}{2}\left(\frac{4}{s^2} - \left(\frac{4}{s^2}\right)^2\cdot\frac{1}{2} + \left(\frac{4}{s^2}\right)^3\cdot\frac{1}{3} - \cdots\right) \to 0 \right)$$
$$+ \frac{1}{2}s\ln\left(\frac{s}{\sqrt{s^2+4}}\right) + \cot^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}s\ln s - \frac{1}{4}s\ln(s^2+4) + \cot^{-1}\left(\frac{s}{2}\right)$$

$$= \lim_{s\to 0} \frac{1}{2}\frac{\ln s}{1/s} - \frac{1}{4}(\ln 4)0 + \cot^{-1}(0)$$

$$= \frac{1}{2}\lim_{s\to 0}\frac{1/s}{-1/s^2} + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

42. Show that

$$\int_0^\infty e^{-2t}\left(\frac{2\sin t - 3\sinh t}{t}\right) dt = 2\cot^{-1}(2) + \frac{3}{2}\ln\left(\frac{1}{3}\right)$$

$$= 2\int_0^\infty e^{-2t}\,\frac{\sin t}{t}\,dt \;-3\int_0^\infty e^{-2t}\,\frac{\sinh t}{t}\,dt$$

$$= 2\mathcal{L}\left\{\frac{\sin t}{t}\right\}_{s=2} - 3\mathcal{L}\left\{\frac{\sinh t}{t}\right\}_{s=2}$$

$$= 2\cot^{-1}(2) - 3\int_2^\infty \frac{1}{s^2-1}\,ds$$

$$= 2\cot^{-1}(2) - \left[\frac{3}{2}\ln\left(\frac{s-1}{s+1}\right)\right]_2^\infty$$

$$= 2\cot^{-1}(2) - \cancelto{0}{\frac{3}{2}\ln(1)} + \frac{3}{2}\ln\frac{1}{3}$$

$$= 2\cot^{-1}(2) + \frac{3}{2}\ln\left(\frac{1}{3}\right)$$

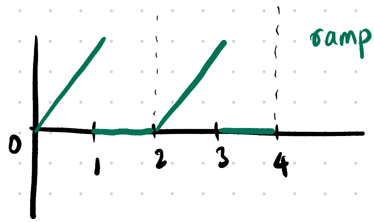
LT of Periodic Functions (proof: self-learning)

If $f(t)$ is a periodic function, i.e. if $f(t+nT) = f(t)$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{sT}} \int_0^T e^{-st} f(t)\,dt \rightarrow \mathcal{L}\{f_1(t)\}$$

(one period)

43.



$$f_1(t) = \begin{cases} kt, & 0 \le t \le 1 \\ 0, & 1 < t \le 2 \end{cases}$$

$$f(t+nT) = f(t)$$

$u = kt \qquad v = -\frac{1}{s} e^{-st}$

$du = k\,dt \qquad dv = e^{-st}$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}} \left(\int_0^1 e^{-st}\, kt\,dt + \int_1^2 e^{-st}\, 0\,dt \right)$$

$$= \frac{1}{1-e^{-sT}} \left(\frac{-kt}{s} e^{-st} \Big|_0^1 + \int_0^1 \frac{k}{s} e^{-st} dt \right)$$

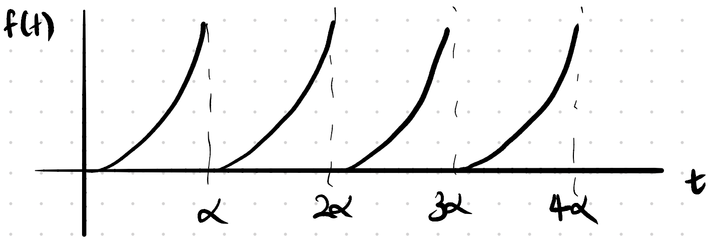
$$= \frac{1}{1-e^{-sT}} \left(\frac{-kT}{s} e^{-sT} - \frac{k}{s^2} e^{-sT} + \frac{k}{s^2} \right)_0^1$$

$$= \frac{K}{1-e^{-sT}} \left(\frac{-T}{s} e^{-sT} - \frac{1}{s^2} e^{-sT} + \frac{1}{s^2} \right)_0^1$$

$$= \frac{K}{1-e^{-2s}} \left(\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right), \quad s \neq 0$$

44. $f_1(t) = t^2;\ 0 < t < \alpha;\ f(t+\alpha) = f(t)$

$\mathcal{L}\{f(t)\} = ?$



$u = t^2$ $\quad v = \frac{-1}{s} e^{-st}$

$du = 2t\,dt$ $\quad dv = e^{-st}\,dt$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{s\alpha}} \int_0^\alpha e^{-st} t^2\,dt$$

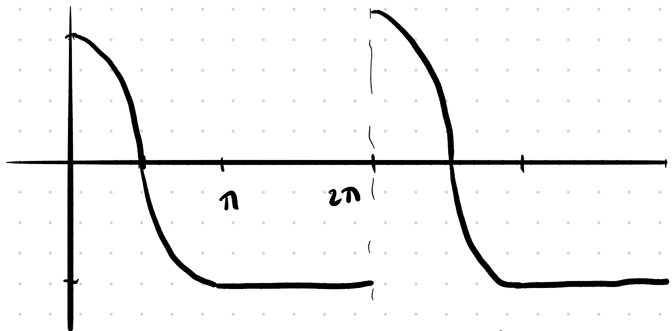
$$= \frac{1}{1-e^{s\alpha}} \left[\frac{-t^2}{s} e^{-st} + \frac{2}{s} \int_0^\alpha t e^{-st}\,dt \right]_0^\alpha$$

$$= \frac{1}{1-e^{s\alpha}} \left[\frac{-\alpha^2}{s} e^{-s\alpha} + \frac{2}{s} \left[\frac{-t}{s} e^{-st} - \int \frac{-1}{s} e^{-st}\,dt \right]_0^\alpha \right]$$

$$= \frac{1}{1-e^{s\alpha}} \left(\frac{-\alpha^2 e^{-s\alpha}}{s} + \frac{2}{s} \left(\frac{-\alpha e^{-s\alpha}}{s} - \frac{1}{s^2} e^{-st} \Big|_0^\alpha \right) \right)$$

$$= \frac{1}{1-e^{s\alpha}} \left(\frac{-\alpha^2 s^{-s\alpha}}{s} - \frac{2\alpha}{s^2} e^{-s\alpha} - \frac{2}{s^3} e^{-s\alpha} + \frac{2}{s^3} \right)$$

45. $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ -1 & \pi \le t \le 2\pi \end{cases}$



$T = 2\pi$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{sT}} \int_0^T e^{-st} f(t)\, dt$$

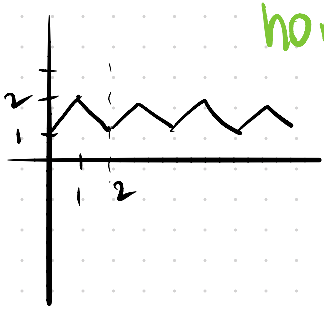
$$= \frac{1}{1-e^{2\pi s}} \left(\int_0^{\pi} e^{-st} \cos t \, dt + \int_{\pi}^{2\pi} -e^{-st} \, dt \right)$$

$$= \frac{1}{1-e^{2\pi s}} \left(\left[\frac{e^{-st}}{s^2+1} (-s\cos t + \sin t) \right]_0^{\pi} + \frac{1}{s} \left[e^{-st} \right]_{\pi}^{2\pi} \right)$$

$$= \frac{1}{1-e^{2\pi s}} \left(\frac{se^{-s\pi}}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s} \left(e^{-2\pi s} - e^{-\pi s} \right) \right)$$

46.

homework



$$f(t) = \begin{cases} 1+t, & 0 \le t < 1 \\ 3-t, & 1 \le t \le 2 \end{cases}$$

$$f(t+2) = f(t) \qquad T = 2$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}} \left[\int_0^1 e^{-st}(1+t)\,dt + \int_1^2 e^{-st}(3-t)\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left[\frac{-e^{-st}}{s}\right]_0^1 + \int_0^1 e^{-st}\,t\,dt - \left[\frac{3}{s} e^{-st}\right]_1^2 - \int_1^2 e^{-st}\,t\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{-e^{-s}}{s} + \frac{1}{s} - \frac{3}{s} e^{-2s} + \frac{3}{s} e^{-s} + \int_0^1 t e^{-st}\,dt - \int_1^2 e^{-st}\,t\,dt \right)$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{2}{s} e^{-s} - \frac{3}{s} e^{-2s} + \frac{1}{s} + \left[\frac{-e^{-st}}{s} t + \frac{e^{-st}}{s^2} \right]_0^1 - \right.$$

$$\left[\frac{-e^{-st}}{s} t + \frac{e^{-st}}{s^2} \right]_1^2$$

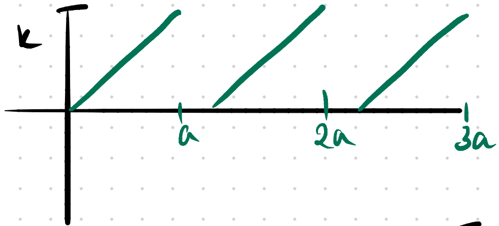
$$= \frac{1}{1-e^{-2s}}\left(\frac{2e^{-s}}{s} - \frac{3e^{-2s}}{s} + \frac{1}{s}\right.$$

$$+ \cancel{\frac{-e^{-s}}{s}} + \cancel{\frac{e^{-s}}{s^2}} - \frac{1}{s^2} + \frac{-e^{-2s}(2)}{s}$$

$$\left. + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \cancel{\frac{e^{-s}}{s^2}}\right)$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{2e^{-s}}{s} - \frac{5e^{-2s}}{s} + \frac{1}{s} - \frac{1}{s^2}\right)$$

47. Find LT of this



$T = a$

$f(t) = \frac{k}{a} t$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{sa}} \int_0^a e^{-st} \frac{k}{a} t\,dt$$

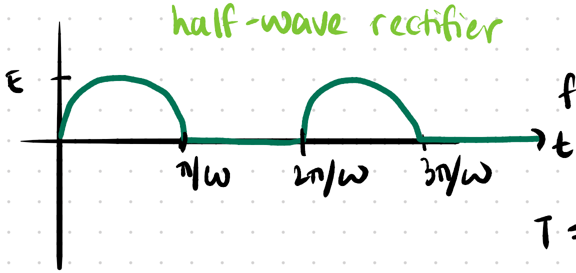
$$= \frac{1}{1-e^{sa}} \frac{k}{a} \left[-\frac{t}{s} e^{-st} + \int_0^a \frac{e^{-st}}{s}\,dt \right]_0^a$$

$$= \frac{1}{1-e^{sa}} \frac{k}{a} \left[-\frac{a}{s} e^{-sa} + -\frac{e^{sa}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{k}{a(1-e^{sa})} \left(\frac{1}{s^2} - \frac{ae^{-sa}}{s} - \frac{e^{sa}}{s^2} \right)$$

48.

half-wave rectifier



$$f(t) = \begin{cases} E\sin\omega t, & 0<t<\pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$$

$T = 2\pi/\omega$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{\pi/\omega} e^{-st} E\sin\omega t\, dt$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left(\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right)\Bigg|_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left(\frac{e^{-s\pi/\omega}\,\omega}{s^2+\omega^2}+\frac{\omega}{s^2+\omega^2}\right)$$

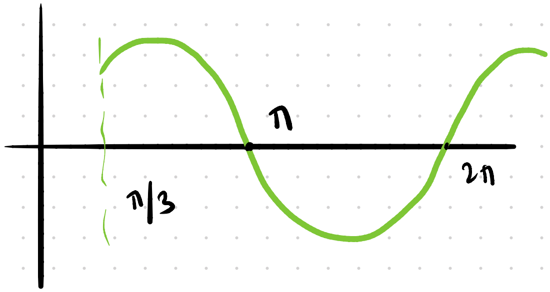
$$= \frac{E\omega\,\cancel{\left(1+e^{-s\pi/\omega}\right)}}{(s^2+\omega^2)\,\cancel{\left(1+e^{-s\pi/\omega}\right)}\left(1-e^{-s\pi/\omega}\right)}$$

$$= \boxed{\frac{E\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}}$$

imp remember

Unit Step Function or Heaviside function or Switch function

electrical engineer

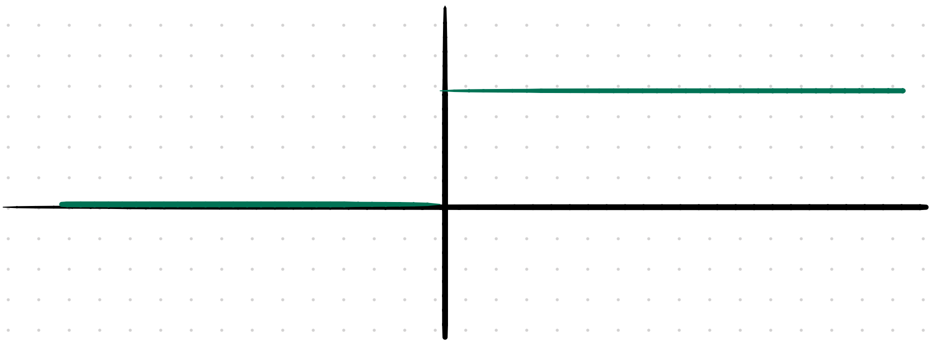


not periodic

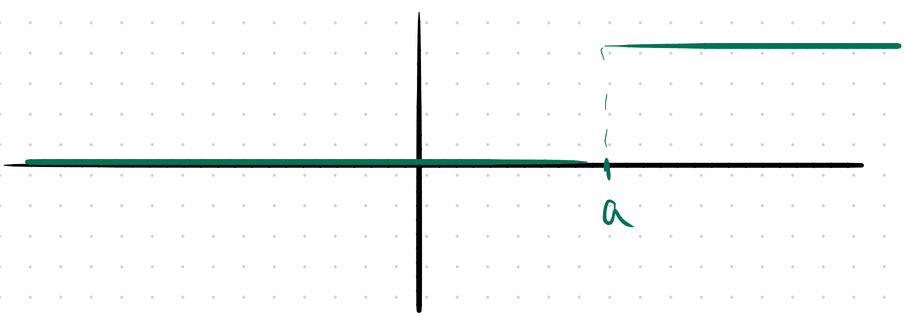
If t is any real number, then $u(t)$ is defined as

$$u(t) = H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Switch: 0 or 1



$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

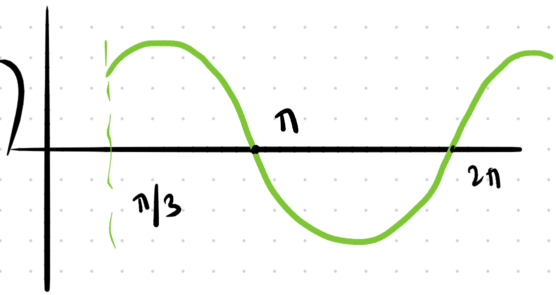


$$\mathcal{L}\{u(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{u(t-a)\} = \int_a^\infty e^{-st}\,dt = \left[\frac{-e^{-st}}{s}\right]_a^\infty$$

$$= \frac{1}{s}\left(e^{-sa}\right) = \frac{e^{-sa}}{s}$$

$\sin t \; u(t-\pi/3)$



$$\mathcal{L}\{\sin t \; u(t-a)\} = ?$$

Second shifting property

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t-a)U(t-a)\} = e^{-as}F(s)$$

Proof

$$\int_0^\infty e^{-st} f(t-a)U(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt \qquad \begin{matrix} v = t-a \\ dv = dt \end{matrix}$$

$$= \int_0^\infty e^{-s(v+a)} f(v)\,dv = e^{-as}\int_0^\infty e^{-sv} f(v)\,dv$$

$$= e^{-as}F(s)$$

Note

$$f(t) = \begin{cases} f_1(t) & 0 < t < a \\ f_2(t) & a \le t < b \\ f_3(t) & t \ge b \end{cases}$$

$$f(t) = f_1(t) + (f_2(t) - f_1(t))\,U(t-a) + (f_3(t) - f_2(t))\,U(t-b)$$

49. $\mathcal{L}\{4\sin(t-3)\mathcal{U}(t-3)\}$

$= 4\mathcal{L}\{f(t-3)\mathcal{U}(t-3)\} = \dfrac{4e^{-3s}}{s^2+1}$

50. $\mathcal{L}\{t^2\mathcal{U}(t-2)\}$

$= \mathcal{L}\{(t-2+2)^2\mathcal{U}(t-2)\}$

$= \mathcal{L}\{((t-2)^2+4(t-2)+4)\mathcal{U}(t-2)\}$

$= f(t) = t^2+4t+4$

$= \mathcal{L}\{(t+2)^2\} = \dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}$

$= e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right)$

51. $\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\} = \mathcal{L}\{-\sin(t-\pi)\,\mathcal{U}(t-\pi)\}$

$= e^{-\pi s}\left(\dfrac{-1}{s^2+1}\right)$ $\qquad = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}$

identify as unit step

for inverse

52. $\mathcal{L}\{(2t-1)\mathcal{U}(t-4)\}$ $\qquad 2(t-4)+7$

$= \mathcal{L}\{(2(t-4)+7)\mathcal{U}(t-4)\}$

$= e^{-4s}\left(\dfrac{2}{s^2} + \dfrac{7}{s}\right)$

53. $\mathcal{L}\{e^{-t}\cos 2t\, u(t-\pi)\}$

$= F(s+1) \Rightarrow F(s) = \mathcal{L}\{\cos 2t\, u(t-\pi)\}$

$F(s) = \mathcal{L}\{-\cos 2(\pi - t)\, u(t-\pi)\}$

$= -\mathcal{L}\{\cos 2(t-\pi)\, u(t-\pi)\}$

$= -e^{-\pi s} \dfrac{s}{s^2+4}$

$\therefore \mathcal{L}\{e^{-t}\cos 2t\, u(t-\pi)\}$

$= \dfrac{-e^{-\pi(s+1)}(s+1)}{(s+1)^2 + 4}$

54. $\mathcal{L}\{e^{-3t} u(t-1)\}$

$= F(s+3) = \dfrac{e^{-(s+3)}}{s+3}$

55. $f(t) = \begin{cases} \sin t & t > \pi \\ \cos t & t < \pi \end{cases} = \begin{cases} \cos t, & t < \pi \\ \sin t, & t > \pi \end{cases}$

not defined

$f(t) = \cancel{(\cos t)\, u(t+\pi) + (\sin t)\, u(t-\pi)}$

$= \cos t + (\sin t - \cos t)\, u(t-\pi)$

$u(t-\pi) = \begin{cases} 0, & t < \pi \\ 1, & t \geq \pi \end{cases}$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + \mathcal{L}\{\sin t \; u(t-\pi)\} - \mathcal{L}\{\cos t \; u(t-\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\frac{(-1)}{s^2+1} - e^{-\pi s}\frac{(-1)\,s}{s^2+1}$$

$$= \frac{s - e^{-\pi s} + s e^{-\pi s}}{s^2+1}$$

for n steps, $n-1$ unit steps

56. Find $\mathcal{L}\{f(t)\}$

$$f(t) = \begin{cases} 2, & 0<t<\pi \\ 0, & \pi \le t < 2\pi \\ \sin t, & t \ge 2\pi \end{cases} \qquad u(t-\pi) = \begin{cases} 0, & t<\pi \\ 1, & t>\pi \end{cases}$$

$$f(t) = 2 + (0-2)\,u(t-\pi) + (\sin t - 0)\,u(t-2\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \mathcal{L}\{\sin t \; u(t-2\pi)\}$$

$$= \frac{2-2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

57. $f(t) = \begin{cases} t^2, & 0<t<2 \\ 4t, & t>2 \end{cases} \qquad u(t-2) = \begin{cases} 0, & t<2 \\ 1, & t>2 \end{cases}$

$$f(t) = t^2 + (4t - t^2)\,u(t-2)$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + 4e^{-2s}\mathcal{L}\{t+2\} - e^{-2s}\mathcal{L}\{(t+2)^2\}$$

$$= \frac{2}{s^3} + 4e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

$$= \frac{2}{s^3} + e^{-2s}\left(\cancel{\frac{4}{s^2}} + \frac{8}{s} - \frac{2}{s^3} - \cancel{\frac{4}{s^2}} - \frac{4}{s}\right)$$

$$= \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

58. Using LT evaluate $\int_0^\infty e^{-t}(1+2t-t^2+t^3)\,u(t-1)\,dt$

$$= \mathcal{L}\{(1+2t-t^2+t^3)\,u(t-1)\}_{s=1}$$

$$(t-1)^3 = t^3 - 3t^2 + 3t - 1$$

$$2(t-1)^2 = +2(t^2-2t+1) = 2t^2 - 4t + 2$$

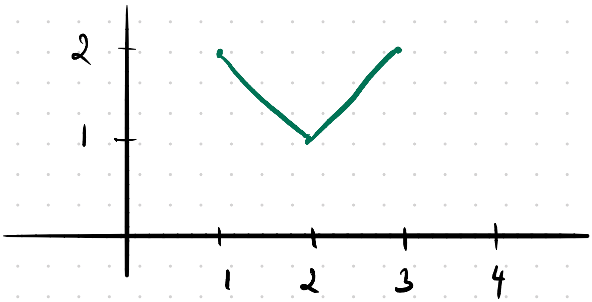
$$3(t-1) = 3t - 3$$

$$+3 = 3$$

$$= e^{-s}\left(\frac{6}{s^4} + \frac{2\times 2}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right)\Bigg|_{s=1}$$

$$= e^{-1}(6+4+3+3) = \frac{16}{e}$$

59. $f(t) = \begin{cases} 3-t & 1 \le t < 2 \\ t-1 & 2 < t < 3 \end{cases}$



needs redefining

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ 3-t, & 1 < t < 2 \\ t-1, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$$

$$f(t) = (3-t)u(t-1) + \left((t-1)-(3-t)\right)u(t-2) + (-t+1)u(t-3)$$

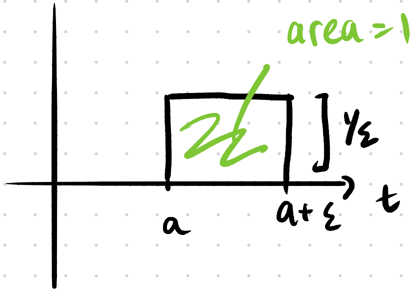
$$= (2-(t-1))u(t-1)$$

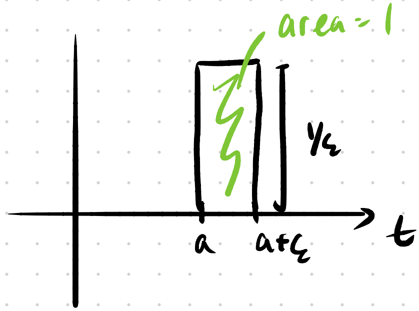
$$+ 2(t-2)u(t-2)$$

$$+ (-(t-3)-2)u(t-3)$$

$$\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{2}{s} - \frac{1}{s^2}\right) + 2e^{-2s}\left(\frac{1}{s^2}\right) + e^{-3s}\left(\frac{-1}{s^2} - \frac{2}{s}\right)$$

Laplace Transform of a Unity Impulse function





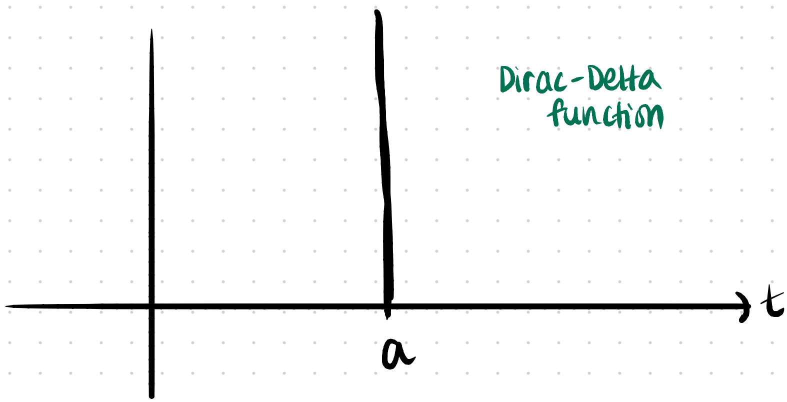
$$\delta_\varepsilon(t-a) = \begin{cases} 0, & 0 < t < a \\ 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & t > a+\varepsilon \end{cases}$$

An impulse is a very large force acting for a short period of time.

It is denoted by $\delta(t-a)$

Geometrically, $\delta(t-a)$ is

$$\underset{\text{(impulse)}}{\delta(t-a)} = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$$



$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

$$u(t-(a+\varepsilon)) = \begin{cases} 0, & t < a+\varepsilon \\ 1, & t \geq a+\varepsilon \end{cases}$$

$$\mathcal{L}\{\delta_\varepsilon(t-a)\} = \mathcal{L}\left\{\frac{1}{\varepsilon} u(t-a) + \frac{-1}{\varepsilon} u(t-(a+\varepsilon))\right\}$$

$$\mathcal{L}\{\delta_\varepsilon(t-a)\} = \frac{1}{\varepsilon}\frac{e^{-as}}{s} - \frac{1}{\varepsilon}\frac{e^{-(a+\varepsilon)s}}{s}$$

$$= \frac{e^{-as}}{s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right)$$

$$\mathcal{L}\{\delta(t-a)\} = \lim_{\varepsilon \to 0} \mathcal{L}\{\delta_\varepsilon(t-a)\}$$

$$= \lim_{\varepsilon \to 0} \frac{e^{-as}}{s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right) \quad \left(\frac{0}{0}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{e^{-as}}{s}\left(\frac{se^{-\varepsilon s}}{1}\right)$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

Properties of Impulse function

1. $\mathcal{L}\{\delta(t-a)\} = \mathcal{L}\{u'(t-a)\}$

$$= sF(s) - f(0) = s\mathcal{L}\{u(t-a)\} - u(0-a)$$

$$= \cancel{s}\frac{e^{-as}}{\cancel{s}} - \cancel{u(-a)}^{\,0}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

2. $\mathcal{L}\{\delta(t)\} = 1$

3. $\displaystyle\int_0^\infty f(t)\,\delta(t-a)\,dt = \lim_{\varepsilon\to 0}\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,dt$

$$\delta_\varepsilon(t-a) = \begin{cases} 0, & 0 < t < a \\ 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & t > a+\varepsilon \end{cases}$$

$$= \lim_{\varepsilon\to 0}\int_0^\infty \delta_\varepsilon(t-a)\,f(t)\,dt$$

$$= \lim_{\varepsilon\to 0}\int_a^{a+\varepsilon} f(t)\,\frac{1}{\varepsilon}\,dt = \lim_{\varepsilon\to 0}\frac{\int_a^{a+\varepsilon} f(t)\,dt}{\varepsilon} \sim \frac{0}{0}$$

$$= \lim_{\varepsilon\to 0}\frac{G(a+\varepsilon) - G(a)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{G'(a+\varepsilon)}{1} = f(a)$$

4. $\mathcal{L}\{f(t)\delta(t-a)\} = \lim_{\varepsilon\to 0} \mathcal{L}\{f(t)\,\delta_\varepsilon(t-a)\}$

$$\delta_\varepsilon(t-a) = \begin{cases} 0 , & 0<t<a \\ 1/\varepsilon , & a<t<a+\varepsilon \\ 0 , & t>a+\varepsilon \end{cases}$$

$$= \lim_{\varepsilon\to 0} \int_0^\infty e^{-st}\delta_\varepsilon(t-a)\, f(t)\, dt$$

$$= \lim_{\varepsilon\to 0} \int_a^{a+\varepsilon} e^{-st} f(t) \frac{1}{\varepsilon}\, dt = \lim_{\varepsilon\to 0} \frac{\int_a^{a+\varepsilon} e^{-st} f(t)\,dt}{\varepsilon} = \frac{0}{0}$$

$$= \lim_{\varepsilon\to 0} \frac{G'(a+\varepsilon)}{1} = e^{-as} f(a)$$

$$= e^{-as} f(a)$$

5) $\mathcal{L}\{t^n \delta(t-a)\} = (-1)^n(-a)^n\, e^{-as} = a^n e^{-as}$

6) $\mathcal{L}\left\{ \frac{\delta(t-a)}{t^n} \right\} = \int_s^\infty \int_s^\infty \cdots e^{-as} \cdots ds\, ds = \frac{e^{-as}}{a^n}$

7) $\mathcal{L}\left\{ \int_0^t \int_0^t \cdots \delta(t-a)\, dt\, dt \cdots \right\} = \frac{e^{-as}}{s^n}$

60. $\mathcal{L}\{t^3\delta(t-1)\} = e^{-s}$

61. $\mathcal{L}\left\{\dfrac{2\delta(t-1)+6\delta(t-2)}{t}\right\} = 2e^{-s} + \dfrac{6e^{-2s}}{2}$

62. $\mathcal{L}\{t^2\cos at\ \cosh at\ \delta(t-2)\} = e^{-2s}(s^2\cos 2a\cosh 2a)$

63. $\mathcal{L}\{\sinh 2t\,\delta(t-3)\} = e^{-3s}\sinh(6)$

64. $\int_0^\infty \cos 2t\ \delta(t-\pi/4)\,dt = \mathcal{L}\{\cos 2t\,\delta(t-\pi/4)\}_{s=0}$

$= e^{-\frac{\pi}{4}0}\cos \pi/2 = 0$

65. $\mathcal{L}\{(t-1)^2\delta(t-a)\} = e^{-as}(a-1)^2$

66. $\mathcal{L}\{\delta(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s) = e^{-as}$

$f(t-a)\mathcal{U}(t-a)$

$F(s) = \mathcal{L}\{\delta(t)\}$